Analytic factorization of Lie group representations

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Abstract

For every moderate growth representation (π, E) of a real Lie group G on a Fréchet space, we prove a factorization theorem of Dixmier–Malliavin type for the space of analytic vectors E^ω. There exists a natural algebra of superexponentially decreasing analytic functions $\mathcal{A}(G)$, such that $E^\omega = \Pi(\mathcal{A}(G))\ E^\omega$. As a corollary we obtain that E^ω coincides with the space of analytic vectors for the Laplace–Beltrami operator on G.

1 Introduction

Consider a category $\mathcal{C}$ of modules over a nonunital algebra $\mathcal{A}$. We say that $\mathcal{C}$ has the *factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} \ = \ \mathcal{A} \cdot \mathcal{M} \ := \ \operatorname{span}\{a \cdot m \ \mid \ a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

In particular, if $\mathcal{A} \in \mathcal{C}$ this implies $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$.

Let (π, E) be a representation of a real Lie group G on a Fréchet space E. Then the corresponding space of smooth vectors E^∞ is again a Fréchet space. The representation (π, E) induces a continuous action Π of the algebra $C_c^\infty(G)$ of test functions on E given by

$$\Pi(f)v = \int_G f(g)\pi(g)v \ dg \quad (f \in \mathrm{C}_c^\infty(G), v \in E),$$

which restricts to a continuous action on E^∞. Hence the smooth vectors associated to such representations are a $C_c^\infty(G)$–module, and a result by Dixmier and Malliavin [3] states that this category has the factorization property.

In this article we prove an analogous result for the category of analytic vectors.

For simplicity, we outline our approach for a Banach representation (π, E). In this case, the space E^ω of analytic vectors is endowed with a natural inductive limit topology, and gives rise to a representation (π, E^ω). To define an appropriate algebra acting on E^ω, we fix a left–invariant Riemannian metric on G and let d be the associated distance function. The continuous functions $\mathcal{R}(G)$ on G which decay faster than $e^{-nd(g,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a $G \times G$–module under the left–right regular representation. We define $\mathcal{A}(G)$ to be the space of analytic vectors of this action. Both $\mathcal{R}(G)$ and $\mathcal{A}(G)$ form an algebra under convolution, and the action Π of $C_c^\infty(G)$ extends to give E^ω the structure of an $\mathcal{A}(G)$–module.

In this setting, our main theorem says that the category of analytic vectors for Banach representations of G has the factorization property. More generally, we obtain a result for F–representations:

Theorem 1.1. *Let G be a real Lie group and (π, E) an F–representation of G. Then*

$$\mathcal{A}(G) \;=\; \mathcal{A}(G) * \mathcal{A}(G)$$

and

$$E^\omega \;=\; \Pi(\mathcal{A}(G))\; E^\omega \;=\; \Pi(\mathcal{A}(G))\; E.$$

Let us remark that the special case of bounded Banach representations of $(\mathbb{R}, +)$ has been proved by one of the authors in [7].
As a corollary of Theorem 1.1 we obtain that a vector is analytic if and only if it is analytic for the Laplace–Beltrami operator, which generalizes a result of Goodman [5] for unitary representations.
In particular, the theorem extends Nelson's result that $\Pi(\mathcal{A}(G))\; E^\omega$ is dense in E^ω [8]. Gårding had obtained an analogous theorem for the smooth vectors [4]. However, while Nelson's proof is based on approximate units constructed from the fundamental solution $\varrho_t \in \mathcal{A}(G)$ of the heat equation on G by letting $t \to 0^+$, our strategy relies on some more sophisticated functions of the Laplacian.
To prove Theorem 1.1, we first consider the case $G = (\mathbb{R}, +)$. Here the proof is based on the key identity

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1,$$

for the entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z\, \mathrm{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$ on the complex plane [1]. We consider this as an identity for the symbols of the Fourier multiplication operators $\alpha_\varepsilon(i\partial)$, $\beta_\varepsilon(i\partial)$ and $\cosh(i\varepsilon\partial)$. The functions α_ε and β_ε are easily seen to belong to the Fourier image of $\mathcal{A}(\mathbb{R})$, so that $\alpha_\varepsilon(i\partial)$ and $\beta_\varepsilon(i\partial)$ are given by convolution with some $\kappa^\varepsilon_\alpha, \kappa^\varepsilon_\beta \in \mathcal{A}(\mathbb{R})$. For every $v \in E^\omega$ and sufficiently small $\varepsilon > 0$, we may also apply $\cosh(i\varepsilon\partial)$ to the orbit map $\gamma_v(g) = \pi(g)v$ and conclude that

$$(\cosh(i\varepsilon\partial)\;\gamma_v) * \kappa^\varepsilon_\alpha + \gamma_v * \kappa^\varepsilon_\beta = \gamma_v.$$

The theorem follows by evaluating in 0.
Unlike in the work of Dixmier and Malliavin, the rigid nature of analytic functions requires a global geometric approach in the general case. The idea is to refine the functional calculus of Cheeger, Gromov and Taylor [2] for the Laplace-Beltrami operator in the special case of a Lie group. Using this tool, the general proof then closely mirrors the argument for $(\mathbb{R}, +)$.
The article concludes by showing in Section 6 how our strategy may be adapted to solve some related factorization problems.

[1]Some basic properties of these functions and the Gaussian error function erf are collected in the appendix.

2 Basic Notions of Representations

For a Hausdorff, locally convex and sequentially complete topological vector space E we denote by $GL(E)$ the associated group of isomorphisms. Let G be a Lie group. By a *representation* (π, E) of G we understand a group homomorphism $\pi : G \to GL(E)$ such that the resulting action

$$G \times E \to E, \quad (g, v) \mapsto \pi(g)v,$$

is continuous. For a vector $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v,$$

the corresponding continuous orbit map.

If E is a Banach space, then (π, E) is called a *Banach representation*.

Remark 2.1. Let (π, E) be a Banach representation. The uniform boundedness principle implies that the function

$$w_\pi : G \to \mathbb{R}_+, \quad g \mapsto \|\pi(g)\|,$$

is a *weight*, i.e. a locally bounded submultiplicative positive function on G.

A representation (π, E) is called an *F-representation* if

- E is a Fréchet space.
- There exists a countable family of seminorms $(p_n)_{n\in\mathbb{N}}$ which define the topology of E such that for every $n \in \mathbb{N}$ the action $G \times (E, p_n) \to (E, p_n)$ is continuous. Here (E, p_n) stands for the vector space E endowed with the topology induced from p_n.

Remark 2.2. (a) Every Banach representation is an F-representation.

(b) Let (π, E) be a Banach representation and $\{X_n : n \in \mathbb{N}\}$ a basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra of G. Define a topology on the space of smooth vectors E^∞ by the seminorms $p_n(v) = \|d\pi(X_n)v\|$. Then the representation (π, E^∞) induced by π on this subspace is an F-representation (cf. [1]).

(c) Endow $E = C(G)$ with the topology of compact convergence. Then E is a Fréchet space and G acts continuously on E via right displacements in the argument. The corresponding representation (π, E), however, is not an F-representation.

2.1 Analytic vectors

If M is a complex manifold and E is a topological vector space, then we denote by $\mathcal{O}(M, E)$ the space of E-valued holomorphic maps. We remark that $\mathcal{O}(M, E)$ is a topological vector space with regard to the compact-open topology.

Let us denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}_\mathbb{C}$ its complexification. We assume that $G \subset G_\mathbb{C}$ where $G_\mathbb{C}$ is a Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Let us stress

that this assumption is superfluous but simplifies notation and exposition. We denote by $\mathcal{U}_{\mathbb{C}}$ the set of open neighborhoods of $\mathbf{1} \in G_{\mathbb{C}}$.

If (π, E) is a representation, then we call a vector $v \in E$ *analytic* if the orbit map $\gamma_v : G \to E$ extends to a holomorphic map to some GU for $U \in \mathcal{U}_{\mathbb{C}}$. The space of all analytic vectors is denoted by E^ω. We note the natural embedding

$$E^\omega \to \lim_{U \to \{\mathbf{1}\}} \mathcal{O}(GU, E), \quad v \mapsto \gamma_v,$$

and topologize E^ω accordingly.

3 Algebras of superexponentially decaying functions

We wish to exhibit natural algebras of functions acting on F-representations. For that let us fix a left invariant Riemannian metric $\mathbf{g}$ on G. The corresponding Riemannian measure dg is a left invariant Haar measure on G. We denote by $d(g, h)$ the distance function associated to $\mathbf{g}$ (i.e. the infimum of the lengths of all paths connecting g and h) and set

$$d(g) := d(g, \mathbf{1}) \qquad (g \in G)\,.$$

Here are two key properties of $d(g)$, see [4]:

Lemma 3.1. *If $w : G \to \mathbb{R}_+$ is locally bounded and submultiplicative (i.e. $w(gh) \leq w(g)w(h)$), then there exist $c, C > 0$ such that*

$$w(g) \leq Ce^{cd(g)} \qquad (g \in G).$$

Lemma 3.2. *There exists $c > 0$ such that for all $C > c$, $\int e^{-Cd(g)}\, dg < \infty$.*

We introduce the space of *superexponentially decaying continuous functions* on G by

$$\mathcal{R}(G) := \left\{ \varphi \in C(G) \mid \forall n \in \mathbb{N} : \sup_{g \in G} |\varphi(g)|\ e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{R}(G)$ is a Fréchet space which is independent of the particular choice of the metric $\mathbf{g}$. A simple computation shows that $\mathcal{R}(G)$ becomes a Fréchet algebra under convolution

$$\varphi * \psi(g) = \int_G \varphi(x)\ \psi(x^{-1}g)\ dx \qquad (\varphi, \psi \in \mathcal{R}(G), g \in G)\,.$$

We remark that the left-right regular representation $L \otimes R$ of $G \times G$ on $\mathcal{R}(G)$ is an F-representation.

If (π, E) is an F-representation, then Lemma 3.1 and Remark 2.1 imply that

$$\Pi(\varphi)v := \int_G \varphi(g)\ \pi(g)v\ dg \qquad (\varphi \in \mathcal{R}(G), v \in E)$$

defines an absolutely convergent integral. Hence the prescription

$$\mathcal{R}(G) \times E \to E, \quad (\varphi, v) \mapsto \Pi(\varphi)v,$$

defines a continuous algebra action of $\mathcal{R}(G)$ (here continuous refers to the continuity of the bilinear map $\mathcal{R}(G) \times E \to E$).

Our concern is now with the analytic vectors of $(L \otimes R, \mathcal{R}(G))$. We set $\mathcal{A}(G) := \mathcal{R}(G)^{\omega}$ and record that

$$\mathcal{A}(G) = \lim_{U \to \{\mathbf{1}\}} \mathcal{R}(G)_U,$$

where

$$\mathcal{R}(G)_U = \left\{ \varphi \in \mathcal{O}(UGU) \mid \forall Q \Subset U \ \forall n \in \mathbb{N} : \sup_{g \in G} \sup_{q_1, q_2 \in Q} |\varphi(q_1 g q_2)| \ e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{A}(G)$ is a subalgebra of $\mathcal{R}(G)$ and that

$$\Pi(\mathcal{A}(G)) \ E \subset E^{\omega}$$

whenever (π, E) is an F-representation.

4 Some geometric analysis on Lie groups

Let us denote by $\mathcal{V}(G)$ the space of left-invariant vector fields on G. It is common to identify $\mathfrak{g}$ with $\mathcal{V}(G)$ where $X \in \mathfrak{g}$ corresponds to the vector field $\widetilde{X}$ given by

$$(\widetilde{X} f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)) \qquad (g \in G, f \in C^{\infty}(G)).$$

We note that the adjoint of $\widetilde{X}$ on the Hilbert space $L^2(G)$ is given by

$$\widetilde{X}^* = -\widetilde{X} - \operatorname{tr}(\operatorname{ad} X).$$

Note that $\widetilde{X}^* = -\widetilde{X}$ in case $\mathfrak{g}$ is unimodular. Let us fix an an orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ with respect to $\mathbf{g}$. Then the Laplace–Beltrami operator $\Delta = d^* d$ associated to $\mathbf{g}$ is given explicitly by

$$\Delta = \sum_{j=1}^{n} (-\widetilde{X_j} - \operatorname{tr}(\operatorname{ad} X_j)) \ \widetilde{X_j}.$$

As $(G, \mathbf{g})$ is complete, Δ is essentially selfadjoint. We denote by

$$\sqrt{\Delta} = \int \lambda \ dP(\lambda)$$

the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define

$$f(\sqrt{\Delta}) = \int f(\lambda) \ dP(\lambda)$$

as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain

$$D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid \int |f(\lambda)|^2 \ d\langle P(\lambda)\varphi, \varphi \rangle < \infty \right\}.$$

Let $c, \vartheta > 0$. We are going to apply the above calculus to functions in the space

$$\mathcal{F}_{c,\vartheta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)| \; e^{c|z|} < \infty \right\},$$
$$\mathcal{W}_{N,\vartheta} = \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < N \} \cup \{ z \in \mathbb{C} \mid |\Im z| < \vartheta |\operatorname{Re} z| \}.$$

The resulting operators are bounded on $L^2(G)$ and given by a symmetric and left invariant integral kernel $K_f \in C^\infty(G \times G)$. Hence there exists a convolution kernel $\kappa_f \in C^\infty(G)$ with $\kappa_f(x) = \kappa_f(x^{-1})$ such that $K_f(x, y) = \kappa_f(x^{-1}y)$, and for all $x \in G$:

$$f(\sqrt{\Delta}) \; \varphi = \int_G K_f(x, y) \; \varphi(y) \; dy = \int_G \kappa_f(y^{-1}x) \; \varphi(y) \; dy = (\varphi * \kappa_f)(x).$$

A theorem by Cheeger, Gromov and Taylor [2] describes the global behavior:

Theorem 4.1. *Let $c, \vartheta > 0$ and $f \in \mathcal{F}_{c,\vartheta}$ even. Then $\kappa_f \in \mathcal{R}(G)$.*

We are going to need an analytic variant of their result.

Theorem 4.2. *Under the assumptions of the previous theorem: $\kappa_f \in \mathcal{A}(G)$.*

Proof. We only have to establish local regularity, as the decay at infinity is already contained in [2].
The Fourier inversion formula allows to express κ_f as an integral of the wave kernel:

$$\kappa_f(\cdot) = K_f(\cdot, \mathbf{1}) = f(\sqrt{\Delta}) \; \delta_{\mathbf{1}} = \int_{\mathbb{R}} \hat{f}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \delta_{\mathbf{1}} \; d\lambda.$$

As we would like to employ $\| \cos(\lambda\sqrt{\Delta}) \|_{\mathcal{L}(L^2(G))} \leq 1$, we cut off a fundamental solution of Δ^k to write

$$\delta_{\mathbf{1}} = \Delta^k \varphi + \psi$$

for a fixed $k > \frac{1}{4} \dim(G)$ and some compactly supported $\varphi, \psi \in L^2$. Hence,

$$\Delta^l \kappa_f(\cdot) = \int_{\mathbb{R}} \hat{f}^{(2k+2l)}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \varphi \; d\lambda + \int_{\mathbb{R}} \hat{f}^{(2l)}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \psi \; d\lambda.$$

In the appendix we show the following inequality for all $n \in \mathbb{N}$ and some constants $C_n, R > 0$

$$|\hat{f}^{(l)}(\lambda)| \leq C_n \; l! \; R^l e^{-n|\lambda|}.$$

Using $\| \cos(\lambda\sqrt{\Delta}) \|_{\mathcal{L}(L^2(G))} \leq 1$ and the Sobolev inequality, we obtain

$$|\Delta^l \kappa_f(\cdot)| \leq C_1 \; (2l)! \; S^{2l}$$

for some $S > 0$. A classical result by Goodman [10] now implies the right analyticity of κ_f, while left analyticity follows from $\kappa_f(x) = \kappa_f(x^{-1})$. Browder's theorem (Theorem 3.3.3 in [6]) then implies joint analyticity. □

4.1 Regularized distance function

In the last part of this section we are going to discuss a holomorphic regularization of the distance function. Later on this will be used to construct certain holomorphic replacements for cut-off functions.
Consider the time–1 heat kernel $\varrho := \kappa_{e^{-\lambda^2}}$ and define $\tilde{d}$ on G by

$$\tilde{d}(g) := e^{-\Delta} d(g) = \int_G \varrho(x^{-1}g)\ d(x)\ dx.$$

Lemma 4.3. *There exist $U \in \mathcal{U}_\mathbb{C}$ and a constant $C_U > 0$ such that $\tilde{d} \in \mathcal{O}(GU)$ and for all $g \in G$ and all $u \in U$*

$$|\tilde{d}(gu) - d(g)| \leq C_U.$$

Proof. According to Theorem 4.2 the heat kernel ϱ admits an analytic continuation to a superexponentially decreasing function on GU for some bounded $U \in \mathcal{U}_\mathbb{C}$. This allows to extend $\tilde{d}$ to GU. To prove the inequality, we consider the integral

$$\bar{\varrho}(y) = \int_G \varrho(x^{-1}y)\ dx$$

as a holomorphic function of $y \in GU$. By the left invariance of the Haar measure and the normalization of the heat kernel, $\bar{\varrho} = 1$ on G, and hence on GU. Recall the triangle inequality on G: $|d(x) - d(g)| \leq d(x^{-1}g)$. This implies the uniform bound

$$\begin{aligned}
\left|\tilde{d}(gu) - d(g)\right| &= \left|\int_G \varrho(x^{-1}gu)\ (d(x) - d(g))\ dx\right| \\
&\leq \int_G \left|\varrho(x^{-1}gu)\right|\ d(x^{-1}g)\ dx \\
&\leq \sup_{v \in U} \int_G \left|\varrho(x^{-1}v)\right|\ d(x^{-1})\ dx.
\end{aligned}$$

□

5 Proof of the Factorization Theorem

Let (π, E) be a representation of G on a sequentially complete locally convex Hausdorff space and consider the Laplacian as an element

$$\Delta = \sum_{j=1}^{n} (-X_j - \operatorname{tr}(\operatorname{ad} X_j))\ X_j$$

of the universal enveloping algebra of $\mathfrak{g}$. A vector $v \in E$ will be called Δ-*analytic*, if there exists $\varepsilon > 0$ such that for all continuous seminorms p on E one has

$$\sum_{j=0}^{\infty} \frac{\varepsilon^j}{(2j)!}\ p(\Delta^j v) < \infty\,.$$

Lemma 5.1. *Let E be a sequentially complete locally convex Hausdorff space and $\varphi \in \mathcal{O}(U, E)$ for some $U \in \mathcal{U}_{\mathbb{C}}$. Then there exists $R = R(U) > 0$ such that for all continuous semi-norms p on E there exists a constant C_p such that*

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}}\varphi\right)(\mathbf{1})\right) \leq C_p\ k!\ R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$.

Proof. There exists a small neighborhood of 0 in $\mathfrak{g}$ in which the mapping

$$\Phi : \mathfrak{g} \to E,\ X \mapsto \varphi(\exp(X)),$$

is analytic. Let $X = t_1 X_1 + \cdots + t_n X_n$. Because E is sequentially complete, Φ can be written for small X and t as

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \left(\widetilde{X_{\alpha_1}} \cdots \widetilde{X_{\alpha_k}}\varphi\right)(\mathbf{1})\, t^{\alpha}.$$

As this series is absolutely summable, there exists a $R > 0$ such that for every continuous semi-norm p on E there is a constant C_p with

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}}\varphi\right)(\mathbf{1})\right) \leq C_p\ k!\ R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$. □

As a consequence we obtain:

Lemma 5.2. *Let (π, E) be a representation of G on some sequentially complete locally convex Hausdorff space E. Then analytic vectors are Δ–analytic.*

In Corollary 5.6 we will see that the converse holds for F-representations.

Let (π, E) be an F-representation of G. Then for each $n \in \mathbb{N}$ there exists $c_n, C_n > 0$ such that

$$\|\pi(g)\|_n \leq C_n \cdot e^{c_n d(g)} \qquad (g \in G),$$

where

$$\|\pi(g)\|_n := \sup_{\substack{p_n(v) \leq 1 \\ v \in E}} p_n(\pi(g)v)\,.$$

For $U \in \mathcal{U}_{\mathbb{C}}$ and $n \in \mathbb{N}$ we set

$$\mathcal{F}_{U,n} = \left\{ \varphi \in \mathcal{O}(GU, E) \mid \forall Q \Subset U\ \forall \varepsilon > 0 :\ \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq))\ e^{-(c_n + \varepsilon) d(g)} < \infty \right\}.$$

We are also going to need the subspace of superexponentially decaying functions in $\bigcap_n \mathcal{F}_{U,n}$:

$$\mathcal{R}(GU, E) = \left\{ \varphi \in \mathcal{O}(GU, E) \mid \forall Q \Subset U\ \forall n, N \in \mathbb{N} :\ \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq))\ e^{N d(g)} < \infty \right\}.$$

We record:

Lemma 5.3. *If $\kappa \in \mathcal{A}(G)_V$, then right convolution with κ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

We denote by $\mathcal{C}_\varepsilon$ the power series expansion $\sum_{j=0}^\infty \frac{\varepsilon^{2j}}{(2j)!}\Delta^j$ of $\cosh(\varepsilon\sqrt{\Delta})$. Note the following consequence of Lemma 5.1:

Lemma 5.4. *Let $U, V \in \mathcal{U}_\mathbb{C}$ such that $V \Subset U$. Then there exists $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon$ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

As in the Appendix, consider the functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \operatorname{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$, which belong to the space $\mathcal{F}_{2\varepsilon,\vartheta}$. We would like to substitute $\sqrt{\Delta}$ into our key identity (A.3)

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1$$

and replace the hyperbolic cosine by its Taylor expansion.

Lemma 5.5. *Let $U \in \mathcal{U}_\mathbb{C}$. Then there exist $\varepsilon > 0$ and $V \subset U$ such that for any $\varphi \in \mathcal{F}_{U,n}$, $n \in \mathbb{N}$,*

$$\mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi$$

holds as functions on GV.

Proof. Note that $\kappa_\alpha^\varepsilon, \kappa_\beta^\varepsilon \in \mathcal{A}(G)$ according to Theorem 4.2. We first consider the case $E = \mathbb{C}$ and $\varphi \in L^2(G)$. With $|\alpha_\varepsilon(z)\cosh(\varepsilon z)|$ being bounded, $\cosh(\varepsilon\sqrt{\Delta})$ maps its domain into the domain of $\alpha_\varepsilon(\sqrt{\Delta})$, and the rules of the functional calculus ensure

$$\varphi - \beta_\varepsilon(\sqrt{\Delta})\varphi = (\alpha_\varepsilon(\cdot)\cosh(\varepsilon\cdot))(\sqrt{\Delta})\varphi = (\cosh(\varepsilon\sqrt{\Delta})\varphi) * \kappa_\alpha^\varepsilon$$

in $L^2(G)$ for all $\varphi \in D(\cosh(\varepsilon\sqrt{\Delta}))$. For such φ, the partial sums of $\mathcal{C}_\varepsilon\varphi$ converge to $\cosh(\varepsilon\sqrt{\Delta})\varphi$ in $L^2(G)$, and hence almost everywhere. Indeed,

$$\begin{aligned}
&\left\| \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^N \frac{\varepsilon^{2j}}{(2j)!}\Delta^j\varphi \right\|_{L^2(G)}^2 \\
&= \int \left\langle dP(\lambda)\left(\cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^N \frac{\varepsilon^{2j}}{(2j)!}\ \Delta^j\varphi\right), \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{k=0}^N \frac{\varepsilon^{2k}}{(2k)!}\ \Delta^k\varphi \right\rangle \\
&= \int \left(\cosh(\varepsilon\lambda) - \sum_{k=0}^N \frac{(\varepsilon\lambda)^{2k}}{(2k)!}\right)^2 \ \langle dP(\lambda)\varphi, \varphi\rangle \\
&= \sum_{j,k=N+1}^\infty \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!}\ \frac{(\varepsilon\lambda)^{2k}}{(2k)!}\ \langle dP(\lambda)\varphi,\varphi\rangle\ ,
\end{aligned}$$

and the right hand side tends to 0 for $N \to \infty$, because

$$\sum_{j,k=0}^\infty \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!}\ \frac{(\varepsilon\lambda)^{2k}}{(2k)!}\ \langle dP(\lambda)\varphi,\varphi\rangle = \int \cosh(\varepsilon\lambda)^2 \langle dP(\lambda)\varphi,\varphi\rangle < \infty\ .$$

In particular, given $\varphi \in \mathcal{R}(GU,E)$ and $\lambda \in E'$, we obtain $\mathcal{C}_\varepsilon\lambda(\varphi) = \cosh(\varepsilon\sqrt{\Delta})\lambda(\varphi)$ almost everywhere and

$$\mathcal{C}_\varepsilon(\lambda(\varphi)) * \kappa_\alpha^\varepsilon + \lambda(\varphi) * \kappa_\beta^\varepsilon = \lambda(\varphi)$$

as analytic functions on G for sufficiently small $\varepsilon > 0$.

Since the above identity holds for all $\lambda \in E'$, we obtain

$$\mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi$$

on any connected domain GV, $\mathbf{1} \in V \subset U$, on which the left hand side is holomorphic.

Recall the regularized distance function $\tilde{d}(g) = e^{-\Delta}d(g)$ from Lemma 4.3, and set $\chi_\delta(g) := e^{-\delta\tilde{d}(g)^2}$ $(\delta > 0)$. Given $\varphi \in \mathcal{F}_{U,n}$, $\chi_\delta\varphi \in \mathcal{R}(GU,E)$ and

$$\mathcal{C}_\varepsilon(\chi_\delta\varphi) * \kappa_\alpha^\varepsilon + (\chi_\delta\varphi) * \kappa_\beta^\varepsilon = \chi_\delta\varphi\,.$$

The limit $\chi_\delta\varphi \to \varphi$ in $\mathcal{F}_{U,n}$ as $\delta \to 0$ is easily verified. From Lemma 5.3 we also get $(\chi_\delta\varphi) * \kappa_\beta^\varepsilon \to \varphi * \kappa_\beta^\varepsilon$ as $\delta \to 0$. Finally Lemma 5.3 and Lemma 5.4 imply

$$\mathcal{C}_\varepsilon(\chi_\delta\varphi) * \kappa_\alpha^\varepsilon \to\ \mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon \quad (\delta \to 0).$$

The assertion follows. □

Proof of Theorem 1.1. Given $v \in E^\omega$, the orbit map γ_v belongs to $\bigcap_n \mathcal{F}_{U,n}$ for some $U \in \mathcal{U}_\mathbb{C}$. Applying Lemma 5.5 to the orbit map and evaluating at $\mathbf{1}$ we obtain the desired factorization

$$v = \gamma_v(\mathbf{1}) = \Pi(\kappa_\alpha^\varepsilon)\,(\mathcal{C}_\varepsilon(\gamma_v)(\mathbf{1})) + \Pi(\kappa_\beta^\varepsilon)\,(\gamma_v(\mathbf{1}))\,.$$

□

Note the following generalization of a theorem by Goodman for unitary representations [5, 10].

Corollary 5.6. *Let* (π, E) *be an F-representation. Then every* Δ*–analytic vector is analytic.*

Remark 5.7. a) A further consequence of our Theorem 1.1 is a simple proof of the fact that the space of analytic vectors for a Banach representation is complete.
b) We can also substitute $\sqrt{\Delta}$ into Dixmier's and Malliavin's presentation of the constant function 1 on the real line [3]. This invariant refinement of their argument shows that the smooth vectors for a Fréchet representation are precisely the vectors in the domain of Δ^k for all $k \in \mathbb{N}$.

6 Related Problems

We conclude this article with a discussion of how our techniques can be modified to deal with a number of similar questions.
In the context of the introduction, given a nonunital algebra $\mathcal{A}$, a category $\mathcal{C}$ of $\mathcal{A}$–modules is said to have the *strong factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} = \{am \ \mid\ a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

6.1 A Strong Factorization of Test Functions

Our methods may be applied to solve a related strong factorization problem for test functions. On $\mathbb{R}^n$ the Fourier transform allows to write a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ as the convolution $\psi * \Psi$ of two Schwartz functions, and [9] posed the natural problem whether one could demand $\psi, \Psi \in \mathcal{R}(\mathbb{R}^n)$. We are going to prove this in a more general setting.

Theorem 6.1. *For every real Lie group G*

$$C_c^\infty(G) \subset \{\psi * \Psi \ \mid \ \psi, \Psi \in \mathcal{R}(G)\}.$$

As above, we first regularize an appropriate distance function and set

$$l(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * \log(1 + |z|).$$

Lemma 6.2. *The function $l(z)$ is entire and approximates* $\log(1+|z|)$ *in the sense that for all $N > 0$, $\vartheta \in (0,1)$ there exists a constant $C_{N,\vartheta}$ such that*

$$|l(z) - \log(1 + |z|)| \leq C_{N,\vartheta} \quad (z \in \mathcal{W}_{N,\vartheta}).$$

Let $m \in \mathbb{N}$. We would like to substitute the square root of the Laplacian associated to a left invariant metric G into a decomposition

$$1 = \widehat{\psi}_m(z)\ \widehat{\Psi}_m(z)$$

of the identity. In the current situation we use $\widehat{\psi}_m(z) = e^{-ml(z)}$ and $\widehat{\Psi}_m(z) = e^{ml(z)}$. Denote the convolution kernels of $\widehat{\psi}_m(\sqrt{\Delta})$ and $\widehat{\Psi}_m(\sqrt{\Delta})$ by ψ_m resp. Ψ_m. The ideas from the proof of Theorem 4.2 may be combined with the results of [2] to obtain:

Lemma 6.3. *Let $\chi \in C_c^\infty(G)$ with $\chi = 1$ in a neighborhood of $\mathbf{1}$. Then $\chi\Psi_m$ is a compactly supported distribution of order m and $(1-\chi)\Psi_m \in \mathcal{R}(G) \cap C^\infty(G)$. Given $k \in \mathbb{N}$, $\psi_m \in \mathcal{R}(G) \cap C^k(G)$ for sufficiently large m.*

Therefore $\widehat{\Psi}_m(\sqrt{\Delta})$ maps $C_c^\infty(G)$ to $\mathcal{R}(G)$. The functional calculus leads to a factorization

$$\mathrm{Id}_{C_c^\infty(G)} = \widehat{\psi}_m(\sqrt{\Delta})\ \widehat{\Psi}_m(\sqrt{\Delta})$$

of the identity, and in particular for any $\varphi \in C_c^\infty(G)$,

$$\varphi = (\widehat{\Psi}_m(\sqrt{\Delta})\ \varphi) * \psi_m \in \mathcal{R}(G) * \mathcal{R}(G).$$

6.2 Strong Factorization of $\mathcal{A}(G)$

It might be possible to strengthen Theorem 1.1 by showing that the analytic vectors have the strong factorization property.

Conjecture 6.4. *For any F-representation (π, E) of a real Lie group G,*

$$E^\omega = \{\Pi(\varphi)v \ \mid \ \varphi \in \mathcal{A}(G),\ v \in E^\omega\}.$$

We provide some evidence in support of this conjecture and verify it for Banach representations of $(\mathbb{R},+)$ using hyperfunction techniques.

Lemma 6.5. *The conjecture holds for every Banach representation of* $(\mathbb{R},+)$.

Proof. Let (π, E) be a representation of $\mathbb{R}$ on a Banach space $(E, \|\cdot\|)$. Then there exist constants $c, C > 0$ such that $\|\pi(x)\| \leq Ce^{c|x|}$ for all $x \in \mathbb{R}$. If $v \in E^\omega$, there exists $R > 0$ such that the orbit map γ_v extends holomorphically to the strip $S_R = \{z \in \mathbb{C} \mid \operatorname{Im} z \in (-R, R)\}$. Let

$$\begin{aligned} \mathcal{F}_+ (\gamma_v)(z) &= \int_{-\infty}^{0} \gamma_v(t) e^{-itz}\, dt, \quad \operatorname{Im} z > c, \\ -\mathcal{F}_- (\gamma_v)(z) &= \int_{0}^{\infty} \gamma_v(t) e^{-itz}\, dt, \quad \operatorname{Im} z < -c. \end{aligned}$$

Define the Fourier transform $\mathcal{F}(\gamma_v)$ of γ_v by

$$\mathcal{F}(\gamma_v)(x) = \mathcal{F}_+ (\gamma_v)(x + 2ic) - \mathcal{F}_- (\gamma_v)(x - 2ic).$$

Note that $\|\mathcal{F}(\gamma_v)(x)\| e^{r|x|}$ is bounded for every $r < R$. Let $g(z) := \frac{Rz}{2} \operatorname{erf}(z)$ and write $\mathcal{F}(\gamma_v)$ as

$$\mathcal{F}(\gamma_v) = e^{-g} e^{g} \mathcal{F}(\gamma_v) \tag{1}$$

Define the inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))$ for $x \in \mathbb{R}$ by

$$\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))(x) = \int_{\operatorname{Im} t = 2c} \mathcal{F}_+ (\gamma_v)(t) e^{itx}\, dt - \int_{\operatorname{Im} t = -2c} \mathcal{F}_- (\gamma_v)(t) e^{itx}\, dt.$$

Applying the inverse Fourier transform to both sides of (1) and evaluating at 0 yields

$$v = (2\pi)^{-1} \Pi \left(\mathcal{F}^{-1}\left(e^{-g}\right)\right) \left(\mathcal{F}^{-1}\left(e^{g}\mathcal{F}(\gamma_v)\right)(0)\right).$$

The assertion follows because $\mathcal{F}^{-1}(e^{-g}) \in \mathcal{A}(\mathbb{R})$. $\square$

Strong factorization likewise holds for Banach representations of $(\mathbb{R}^n,+)$. Using the Iwasawa decomposition we are able to deduce from this the conjecture for $SL_2(\mathbb{R})$.

A An Identity of Entire Functions

Consider the following space of exponentially decaying holomorphic functions

$$\begin{aligned} \mathcal{F}_{c,\vartheta} &= \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)|\, e^{c|z|} < \infty \right\}, \\ \mathcal{W}_{N,\vartheta} &= \{z \in \mathbb{C} \mid |\Im z| < N\} \cup \{z \in \mathbb{C} \mid |\Im z| < \vartheta |\Re z|\}. \end{aligned}$$

To understand the convolution kernel of a Fourier multiplication operator on $L^2(\mathbb{R})$ with symbol in $\mathcal{F}_{c,\vartheta}$, or more generally functions of $\sqrt{\Delta}$ on a manifold as in Section 4, we need some properties of the Fourier transformed functions.

Lemma A.1. *Given $f \in \mathcal{F}_{c,\vartheta}$, there exist $C, R > 0$ such that*

$$|\hat{f}^{(k)}(z)| \leq C_n \; k! \; R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$.

Proof. Given $f \in \mathcal{F}_{c,\vartheta}$, the Fourier transform extends to a superexponentially decaying holomorphic function on $\mathcal{W}_{c,\vartheta}$. It follows from Cauchy's integral formula that

$$|\hat{f}^{(k)}(z)| \leq C_n \; k! \; R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$. □

Some important examples of functions in $\mathcal{F}_{c,\vartheta}$ may be constructed with the help of the Gaussian error function [11]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} \; dt.$$

The error function extends to an odd entire function, and $\operatorname{erf}(z) - 1 = O(z^{-1} e^{-z^2})$ as $z \to \infty$ in a sector $\{|\operatorname{Im} z| < \vartheta \operatorname{Re} z\}$ around $\mathbb{R}_+$.

Remark A.2. The function

$$z \operatorname{erf}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * |z| - \frac{1}{\sqrt{\pi}} e^{-z^2}$$

is just one convenient regularization of the absolute value $|z|$, and the basic properties we need also hold for other similarly constructed functions. For example replace the heat kernel $\frac{1}{\sqrt{\pi}} e^{-z^2}$ by a suitable analytic probability density.

For any $\varepsilon > 0$, some algebra shows that the even entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \operatorname{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$ decay exponentially as $z \to \infty$ in $\mathcal{W}_{N,\vartheta}$ for any $\vartheta < 1$. Hence $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$. Our later factorization hinges on a multiplicative decomposition of the constant function 1:

Lemma A.3. *For all $\varepsilon > 0, \vartheta \in (0, 1)$, the functions $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$ satisfy the identity*

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1.$$

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[11] see e.g. http://functions.wolfram.com